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# IDEALIZED SHEATH THEORY AND SATELLITE CHARGE-UP IN THE VAN ALLEN REGION

G. E. Modesitt



PREPARED FOR:
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## PREFACE

This report deals with one aspect of RAND's continuing investigation of the effects of the space environment on artificial earth satellites. The problem dealt with here is the naturally occurring charge-up or electric potential attained by a satellite as it passes through the ionized regions above the earth. The potential may have important effects on the response of instruments in space probes, on radiations inside manned space vehicles, and on satellite lifetimes. Thus, the present report, while of only theoretical interest in itself, should ultimately have application in these areas of space studies.

## SUMMARY

As an aid in the determination of the electric potential of naturally charged satellites, the concept of the idealized sheath introduced by Langmuir and Mott-Smith is studied in some detail through the use of distribution functions. It is shown that the functions are discontinuous in velocity variables and lead to the same results as particle trajectory theory. The limitations of the sheath theory and its connection with space-charge-limited diode theory are discussed. It is shown that, under certain assumptions, the potential on a satellite whose diameter is smaller than the local Debye length will reach 3500 volts negative in the more intense regions of the Van Allen electron belt. The equilibrium potential decreases with increasing size of the satellite, with a limiting value of -35,000 volts for satellites much greater than the Debye length in diameter.

## ACKNOWLEDGMENTS

Many helpful discussions with Carl Greifinger and Paul Tamarkin are gratefully acknowledged. R. O. Hundley kindly communicated some of his results prior to their publication.

# CONTENTS

## SYMBOLS

| | |
|---|---|
| $a$ | sheath radius |
| $E$ | initial value of particle kinetic energy outside sheath |
| $E_c$ | $\mu z$ |
| $e$ | charge on proton |
| $M$ | proton mass |
| $m$ | electron mass |
| $N$ | particle density |
| $n$ | exponent showing energy dependence of Van Allen flux |
| $n(E)$ | isotropic electron density in velocity space |
| $n_p(E)$ | isotropic proton density in velocity space |
| $R$ | satellite radius |
| $T$ | Maxwellian temperature |
| $t$ | proton tangential velocity |
| $u$ | proton radial velocity |
| $v_e$ | mean thermal electron velocity |
| $v_i$ | mean thermal ion velocity |
| $x$ | $-e\,\phi_R/kT$ |
| $z$ | $-e\,\phi_R$ |
| $z_o$ | value of z for zero sheath thickness |
| $z_\infty$ | value of z for infinite sheath |
| $\Gamma$ | total electron flux at satellite |
| $\Gamma_o$ | total electron flux outside sheath |
| $\Gamma_p$ | total proton flux at satellite |
| $\Gamma_{po}$ | total proton flux outside sheath |

| | |
|---|---|
| $\mu$ | $R^2/(a^2 - R^2)$ |
| $\phi_R$ | satellite potential |
| $\psi(E)$ | electron flux above energy E outside sheath |
| $\omega$ | particle kinetic energy |

# I. INTRODUCTION

The initial flow of electrons to an absorbing body placed in a neutral isothermal plasma will exceed the flow of positive ions, and the body will begin to accumulate a net negative charge. As the charge builds up, it will repel more plasma electrons, attract more ions, and the charging rate will tend toward zero.

A space vehicle moving through the plasma generally assumed to exist at and beyond several earth radii from the earth should tend to become negatively charged in a similar manner, although the amount of charge accumulated may be considerably modified by solar photoeffects and secondary emission. The intense flux of geomagnetically trapped electrons in the outer radiation belt should produce an unusually large negative charge on a vehicle penetrating this region.

Solar coronal theories indicate that the interplanetary gas in the region of the earth's orbit consists primarily of protons and electrons at temperatures of at least $20,000^\circ$ K, and that the coronal gas streams out from the sun with its stream velocity large compared with the proton mean thermal speed. This solar wind can normally penetrate the sunward side of the earth's magnetic field to a distance of about fifteen earth radii from the earth's center. The condition of the gas within the shielded region (the magnetosphere) around the earth is not well known. Chapman,[1] for example, suggests the temperature above the earth's atmosphere rises gradually to the outside value, whereas Johnson proposes the temperature throughout the magnetosphere is relatively constant at about $1250^\circ$ K.[2] Whistler data are reasonably consistent with the higher-temperature model, while satellite ion probe data tend to show much lower

temperatures. Neither model can claim confirmation at this time.

Satellite charge-up in the ionosphere has been treated by Jastrow and Pearse,[3] who assume the satellite potential $\phi_R$ is negative and that the satellite is surrounded by a thin, positively charged sheath. With the satellite velocity intermediate between the ion and electron velocities, they find the ion density around the satellite unchanged from its ambient ionospheric density, whereas the electron density is decreased by the electron Boltzmann factor $e^{-x}$, where

$$x = - \frac{e\phi_R}{kT}$$

Under these conditions, the ion flux at the satellite is equal to the product of the ambient ion density and the satellite velocity $v_s$, and the electron flux is equal to the product of the ambient electron flux and the Boltzmann factor. The ambient flux is one-fourth the product of the ambient density and mean thermal velocity $v_e$, and, when the electron flux equals the ion flux,

$$x = \ln \frac{v_e}{4\ v_s}$$

In their drag calculations, Jastrow and Pearse find that the effective cross section for ion-satellite collisions is about twice the satellite geometric cross section. Their neglect of this effect (which will decrease x by ln 2) in the potential calculation, and their somewhat incorrect form for the electron flux (their Eq. (4) results in a slightly modified Boltzmann factor), will not alter significantly their result that $x \approx 7$.

In contrast, Chopra[4] conjectures that the shielding of the Coulomb potential of the satellite is negligible, from which he concludes that the potential of a body at rest in an ionized gas is given by

$$x = \ln \frac{v_e}{v_i}$$

where $v_i$ is the ion mean thermal velocity. It can be shown (see Section VI following) that the above result for x should be decreased by $\ln(1 + x)$. It is shown in Section VI that Chopra's result is correct for conditions opposite those he assumed, and is applicable to a body completely shielded by a vanishingly thin sheath. For a plasma of electrons and protons at equal temperatures, it is shown that the extreme values are $x = 2.5$ (no shielding) and $x = 3.8$ (complete shielding), and that intermediate shielding gives intermediate values of x.

Although details of the sheath structure are apparently unnecessary for calculation of the potential on the satellite, they have important bearing on drag and other effects. If the sheath is sufficiently thin and the satellite sufficiently attractive to the ions, it is sometimes possible[5] to make order-of-magnitude estimates of the charge density in the sheath with the aid of conventional probe theory. Probe theory in the usual sense, however, often compounds the inconsistencies of the idealized sheath with generally inapplicable diode theory, and is of little general value. More recent probe theories, such as that of Bernstein and Rabinowitz,[6] attempt to determine the particle distribution functions from Boltzmann's equations. Such treatments encounter many mathematical difficulties, however, and no general solutions have been obtained. Ichimiya,

et al.,[7] in their discussion of rocket ion probes, assume (see their Eqs. (1) and (2)) that the ion current to the probe is given both by the Mott-Smith expression[8] based on the ideal sheath condition, and by the expression for the current in a cold spherical diode. Although the latter expression can be shown to be the zero temperature limit of the former (under certain further restrictions; see the discussion at the end of Section IV), they equate the two expressions in order to eliminate the sheath thickness. Similarly, Hundley[9] relies heavily on diode theory to calculate the satellite potential in the Van Allen regions, and finds it necessary to attempt to approximate the cumbersome diode solutions with simpler expressions. However, it will be shown that even in the presence of the Van Allen electrons, potentials calculated on the basis of the ideal sheath theory vary only by a factor of two or three from an intermediate value over the entire range of possible sheath thicknesses. Since the errors from the uncertainties in the characteristics of the ambient plasma and Van Allen flux cause considerably more variation, Hundley's sheath thickness estimates, however incorrect, are largely irrelevant to his estimate of the satellite potential.

It is the purpose of the following to demonstrate the simplicity of the ideal sheath theory in the evaluation of the current to a body at rest in collisionless ionized gas, and to apply these results to the satellite potential in the Van Allen region. In view of the occasional confusion on the subject, and in order to provide a general foundation for the theory, the method of the distribution function and the Boltzmann equation is used. It is shown that certain discontinuous distribution functions lead to the same result as the particle trajectory methods, so that, in particular, the results of Mott-Smith and Langmuir[8] are recovered and extended.

## II. METHOD

In order to estimate the potential on a satellite in the radiation belt, several simplifying assumptions will be made. It will be assumed that the gas is an infinite, homogeneous, collisionless plasma of protons and electrons with isotropic velocity distribution, and that nearly all proton velocities are much greater than the satellite velocity. Since the velocity of a satellite in the outer radiation belt (at about four earth radii from the center of the earth) equals the average proton speed in a Maxwellian distribution at $1000^{\circ}$ K, this assumption is very good for the continuously rising temperature assumption, but poor for the constant low temperature assumption. The high temperature approximation assures spherical symmetry to the problem and permits analytic evaluations of the proton currents to the satellite. Furthermore, the results so obtained will be directly applicable to static floating probe theory. If the satellite and average proton speeds are assumed comparable, the symmetry is lost and numerical methods will be required. Even if the proton temperature is as low as $1250^{\circ}$ K the results based on the high temperature approximation will provide reasonable estimates of the satellite potential, although many of the details of the proton current calculations will be more precise than the approximation warrants.

The perturbing effects of solar photons, etc., will be neglected; the satellite will be assumed to be a conducting sphere which absorbs all incident particles, and the electrons will be treated non-relativistically. Although the only quantity to be calculated is the potential of the satellite, it is of interest to introduce the proton and electron distribution functions and the equations they satisfy, the collisionless (homogeneous)

Boltzmann equations. If the origin of the coordinate system is at the center of the satellite, then, by symmetry, the electron and proton currents must be radial, and each distribution function may be divided into two parts corresponding to inward--or outward--moving particles. Usually, in the steady state, only the inward moving particles will be of interest, and it is convenient to identify the flux of such particles. The expression, "inward electron flux," for example, will refer to the flux of electrons with negative radial velocity components. The radial electron current is the difference between the outward and inward electron flux. In an infinite uniform gas with an isotropic velocity distribution, the flux in any direction is one-fourth the product of the gas density and the mean particle speed, and the current is zero.

One boundary condition on the solutions to the Boltzmann equation, that of the perfectly absorbing sphere, means that the outward electron flux and outward proton flux each vanish at the surface of the sphere. Another boundary condition characterizes the plasma at infinity or, more generally, beyond a particular radial distance (possibly infinite) from the center of the satellite. It will be assumed that at this radial distance, which will be called the sheath edge, the inward proton and inward electron distribution functions are known, and are isotropic in their velocity variables. Essentially, the isotropic assumption implies the usual idealized sheath conditions, i.e., that the potential outside the sheath region vanishes, and that the plasma there is in an unperturbed state. The assumption is therefore contradictory (since in the presence of the satellite there will be radial currents of electrons and protons throughout the entire plasma) for a finite sheath, although hopefully it is a good

approximation if the potential at the edge of the sheath is sufficiently small.

Since the distribution functions vanish inside the absorbing sphere, they satisfy the Boltzmann equations trivially in this region. The surface of the sphere represents a surface of discontinuity in configuration space for the distribution functions, and a surface on which the Boltzmann equation does not apply. Similarly, in the presence of the negative potential, there are regions in phase space where the sum of the proton kinetic and potential energies is negative, and throughout which the proton distribution function vanishes in the absence of collisions. The surface of zero proton total energy is a surface of discontinuity for the proton distribution function, and a surface on which the Boltzmann equation is not satisfied.

Since the incoming (outgoing) electron distribution function vanishes for positive (negative) radial velocities, the zero radial velocity surface in electron velocity space is, in general, a surface of discontinuity for the partial electron distribution functions. In the steady state, the electron partial distribution functions independently satisfy the Boltzmann equation everywhere except on this surface. Similar considerations hold for the proton partial distribution functions, except that the surface of discontinuity is the zero total energy surface plus that section of the zero radial velocity surface which lies in the positive total energy region of proton phase space.

Steady state solutions for proton and electron inward moving distribution functions will be obtained. Each inward moving distribution function vanishes on one side of its surface of discontinuity; on the other side it is a solution of the collisionless Boltzmann equation which satisfies the boundary condition for the region outside the sheath.

## III. THE ELECTRON FLUX

The collisionless Boltzmann equation states that the distribution function is constant along a particle trajectory. Assume the electrostatic potential $\phi_R$ at the surface of the satellite is negative and that $\phi(r) \geq \phi_R$ for all $r > R$, where R is the radius of the satellite. The velocity distribution of inward moving particles is assumed isotropic for $r > a$, where a is the sheath radius. For $r > a$ the electron partial distribution function f is a function only of $\omega$, the kinetic energy. Since the total energy of an electron is conserved along its trajectory, the (inward) distribution function is $f(\omega + z)$, where $z = - e\phi_R$ and is the potential energy of an electron at R.

The flux $\Gamma$ at a point r in a given direction may be calculated from the partial distribution function for particles moving in that direction. With spherical coordinates, the velocity component in the given direction is $v \cos \theta$, and the flux will involve an integration over velocity space of this factor times the partial distribution function. When the velocity distribution is isotropic, the integration over angles may be carried out to give

$$\Gamma = A \int_0^\infty \omega f(\omega, \vec{r})\, d\omega \qquad (1)$$

where A is a constant. In the case of the radially inward moving electrons in the spherically symmetric field, the inward flux at R is given by

$$\Gamma = A \int_0^\infty \omega f(\omega + z)\, d\omega \qquad (2)$$

The inward flux for $r > a$ is given by

$$\Gamma_o = A \int_0^\infty \omega f(\omega) d\omega = \frac{N}{4} v_e \quad . \tag{3}$$

where $\Gamma_o$ is the flux in any direction in the absence of the satellite, N is the density, and $v_e$ the mean speed of the unperturbed electrons.

If the partial distribution function for $r > a$ is Maxwellian, that is, if it goes as $\exp(-\omega/kT)$, then for $r = R$, it is changed by the factor $e^{-x}$ where

$$x = \frac{z}{kT} = - \frac{e\phi_R}{kT} \tag{4}$$

The factor $e^{-x}$ may be brought outside the integral sign in Eq. (2), and it follows immediately that the inward electron flux to the satellite is

$$\Gamma = e^{-x} \Gamma_o \tag{5}$$

Instead of the distribution function, the "integral flux" $\psi$ (E) above the energy E may be given as a boundary condition, where

$$\psi(E) = A \int_E^\infty \omega f(\omega) d\omega \tag{6}$$

and $\psi(0) = \Gamma_o$. From Eq. (6),

$$f(\omega) = - \frac{\psi'(\omega)}{A\omega} \tag{7}$$

where the prime indicates differentiation. Substitution of Eq. (7) into Eq. (2) gives the inward electron flux to the satellite in terms of the unperturbed integral flux as

$$\Gamma = -\int_0^\infty \frac{\omega\psi'(\omega + z)}{\omega + z}\, d\omega$$

$$= -\int_z^\infty \frac{(\omega - z)\psi'(\omega)}{\omega}\, d\omega$$

$$= \psi(z) + z\int_z^\infty \frac{\psi'(\omega)}{\omega}\, d\omega$$

Integration by parts gives

$$\Gamma = z\int_z^\infty \frac{\psi(\omega)}{\omega^2}\, d\omega \tag{8}$$

An example of the integral flux is the monoenergetic source at energy $E_o$, that is, where

$$\psi(E) = \begin{cases} \Gamma_o & E \leq E_o \\ 0 & E > E_o \end{cases}$$

where $\Gamma_o$ is a constant. From Eq. (8) the flux at the satellite becomes

$$\Gamma = \begin{cases} (1 - \frac{z}{E_o})\, \Gamma_o & z \leq E_o \\ 0 & z > E_o \end{cases}$$

For sufficiently high energies, the integral flux in the outer belt of trapped electrons is assumed to decrease inversely as some power of the energy. If, for energies $E \geq z$, the unperturbed flux $\psi(E)$ goes as $E^{-n}$ with $n > 0$, then, from Eq. (8), the flux at the satellite becomes

$$\text{[illegible]} = \frac{\psi(z)}{n + 1} \qquad (9)$$

## IV. THE PROTON FLUX

The (inward) proton distribution function may be treated in the same manner as were the electron functions except for those regions of phase space where the sum of the kinetic energy $\omega$ and the potential energy $e\phi_R = -z$ is less than zero. In these "forbidden" regions where $\omega < z$ the distribution function vanishes, since an originally free particle cannot (in the absence of collisions) move into a region where its total energy is negative.

In order to find the forbidden region of velocity space at the surface of the satellite, consider a proton with radial velocity u (positive inward) and tangential velocity t at the surface. If $u_o$ and $t_o$ are the corresponding velocities at the edge of the sheath, then, by conservation of energy,

$$u^2 + t^2 + \frac{2e\phi_R}{M} = u_o^{\,2} + t_o^{\,2} \tag{10}$$

where M is the proton mass. (The relation in Eq. (10) is a necessary condition for a particle reaching the satellite. The sufficient condition, that is, the condition on the potential in the intermediate region such that all particles satisfying the conservation laws actually reach the surface will be discussed later.) Conservation of angular momentum gives

$$Rt = at_o \tag{11}$$

and $t_o$ may be eliminated from Eqs. (10) and (11) to give

$$u^2 + \left(1 - \frac{R^2}{a^2}\right)t^2 + \frac{2e\phi_R}{M} = u_o^{\,2} \tag{12}$$

If, as in a Maxwellian plasma, a continuous range of initial radial speeds from zero to infinity exists, the distribution function at R vanishes identically if, and only if, the corresponding velocity variables are such that the expression given by the left side of Eq. (12) is negative.

A convenient way to calculate the proton flux at the satellite surface is to assume at first that if the incoming proton distribution function is $f_p(\omega)$ at $r = a$, then it is given by $f_p(\omega - z)$ at $r = R$ for all positive $\omega$, and to subtract the flux associated with regions where $\omega < z$. For a Maxwellian distribution, it follows immediately that the flux of protons is equal to $e^x$ times the difference between the unperturbed values of the total and the "forbidden" flux, that is,

$$\Gamma_p = e^x(\Gamma_{po} - \Gamma_{pf}) \tag{13}$$

where the subscript p refers to protons. The ratio of the forbidden to total flux is

$$\frac{\Gamma_{pf}}{\Gamma_{po}} = \frac{\int_o^{u_1} udu \int_o^{t_1} tdt\, e^{-\frac{M(u^2 + t^2)}{2\,kT}}}{\int_o^{\infty} udu \int_o^{\infty} tdt\, e^{-\frac{M(u^2 + t^2)}{2\,kT}}}$$

where, from Eq. (12), the upper limits for the forbidden region are given by

$$t_1{}^2 = \frac{a^2}{a^2 - R^2}(u_1{}^2 - u^2)$$

$$u_1{}^2 = -\frac{2e\phi_R}{M} = \frac{2z}{M} \tag{14}$$

The integration may be carried out and the results expressed as

$$\Gamma_p = \Gamma_{po}\left[\frac{1 + \mu - e^{-\mu x}}{\mu}\right] \tag{15}$$

where

$$\mu = \frac{R^2}{a^2 - R^2} \tag{16}$$

is a positive parameter characterizing the relative sizes of the satellite and sheath. This result was first obtained on the basis of particle trajectory methods by Mott-Smith and Langmuir.[8] A satellite potential may be regarded as strong if $\mu x \gg 1$, that is, if

$$x = -\frac{e\phi_R}{kT} \gg \frac{a^2 - R^2}{R^2} = \frac{1}{\mu} \tag{17}$$

for then the proton flux at the satellite approaches

$$\Gamma_p \longrightarrow \Gamma_{po}\left(\frac{1 + \mu}{\mu}\right) = \Gamma_{po}\,\frac{a^2}{R^2} \tag{18}$$

This limiting value may also be interpreted as the thin sheath approximation, although strictly it is always necessary to consider both potential and sheath size together. In the opposite limit, $\mu x \ll 1$, that of weak potentials or thick sheath, the flux approaches

$$\Gamma_p \longrightarrow \Gamma_{po}\,(1 + x) = \Gamma_{po}\left[\frac{kT - e\phi_R}{kT}\right] \tag{19}$$

## V. CONSTRAINTS ON POTENTIAL SHAPES

The preceeding results are valid only if the potential throughout the sheath satisfies certain conditions, and under more general conditions the calculations may proceed stepwise. The conditions are the same for all methods based on the use of energy and angular momentum conservation at two separated points, and are imposed to insure that the proper trajectory connecting the two points exists.

The condition on the validity of Eq. (5) for electrons moving in a repulsive field is the rather weak restriction that throughout the sheath region

$$\phi(r) > \phi_R \tag{20}$$

This condition is easily derived when the time-reversed problem of an electron directed from the surface of the satellite into the sheath is considered. If electron speeds down to zero are permitted, then if and only if Eq. (20) holds will all ejected electrons escape the sheath.

The condition for the validity of Eq. (15) for the proton flux may be obtained from the requirement that protons which penetrate the entire sheath do not reach zero radial velocity in the sheath. From the conservation laws this requirement will be met throughout the sheath only if

$$e\phi < \frac{M}{2}(u_o^2 + t_o^2) - \frac{M}{2}\frac{a^2}{r^2}t_o^2 \tag{21}$$

where $\phi$ is the potential at r, and $u_o$ and $t_o$ are the velocity components at a, the sheath edge. If $\phi_R$, u, and t are the corresponding values at R, the surface of the satellite, then

$$\frac{M}{2} u^2 + e\phi_R = \frac{M}{2} u_o^2 + \frac{M}{2} t_o^2 \left(1 - \frac{a^2}{R^2}\right) \tag{22}$$

Equation (22) may be solved for $t_o^2$ and the result substituted into Eq. (21) to give

$$e\phi < \left[\frac{\frac{a^2}{r^2} - 1}{\frac{a^2}{R^2} - 1}\right] e\phi_R$$

$$+ \left[\frac{1 - \frac{R^2}{r^2}}{1 - \frac{R^2}{a^2}}\right] \frac{Mu_o^2}{2} + \left[\frac{\frac{a^2}{r^2} - 1}{\frac{a^2}{R^2} - 1}\right] \frac{Mu^2}{2} \tag{23}$$

As u approaches zero and t approaches $t_1$, given by Eq. (14), $u_o$ also approaches zero, and the two positive terms on the rignt-nand side of Eq. (23) can vanish simultaneously. It is therefore necessary and sufficient that for all $R < r < a$,

$$\phi(r) < \phi_M(r) = \left[\frac{\frac{a^2}{r^2} - 1}{\frac{a^2}{R^2} - 1}\right] \phi_R \tag{24}$$

in order that the proton flux be given by Eq. (15). The limiting potential $\phi_M(r)$ is proportional to $1/r^2$ plus a constant, with the constants determined by $\phi_M(R) = \phi_R$ and $\phi_M(a) = 0$. The restriction of the potential to values less than $\phi_M$ will be called the sheath condition. In the absence

of infinite charge densities, solutions of Poisson's equation which vanish for $r \geq a$ cannot satisfy the sheath condition, but if the point where the potential rises above $\phi_M$ is near the sheath edge the error in Eq. (15) will be correspondingly small.

If the results in Eqs. (5) and (15) are simultaneously valid, the potential must satisfy

$$\phi_R < \phi < \phi_M$$

A more restrictive lower limit may be obtained from the reasonable assumption that the charge density is positive throughout the sheath. If this condition holds, the electric field intensity inside the sheath is negative,

$$\phi' > 0$$

and $\phi$ increases monotonically throughout the sheath. If $\phi_C$ is a potential which varies as $1/r$ plus a constant and satisfies $\phi_C(R) = \phi_R$ and $\phi_C(a) = 0$, then

$$\phi_C'' = - \frac{r}{2} \phi_C' \tag{25}$$

that is, $\phi_C$ represents a "sheath" of zero charge density. In a positively charged sheath it is necessary that $\phi$ satisfy the inequality

$$\phi'' < - \frac{r}{2} \phi'$$

Since $\phi$ is a solution of Poisson's equation, $\phi$ is greater than $\phi_C$ near the

sheath edge, and it can be shown from Eq. (25) that the curves representing $\phi$ and $\phi_C$ cannot cross inside the sheath. For if, as r decreases from the sheath edge, the potential $\phi$ falls below $\phi_C$, its slope must continue to decrease so rapidly in order to satisfy Eq. (25) that $\phi(R)$ will be below $\phi_R$. Hence, everywhere inside the positively charged sheath, the potential $\phi$ must be greater than $\phi_C$. This lower limit on the potential is more restrictive and can replace that given by Eq. (20), and implies that, if the proton flux calculation is valid in a positively charged sheath, the potential inside the sheath is restricted to the rather narrow range

$$\phi_C < \phi < \phi_M$$

## VI. THE SATELLITE POTENTIAL

In a neutral Maxwellian plasma the satellite will attain a potential such that the electron flux in Eq. (5) equals the proton flux in Eq. (15) at the surface, that is,

$$\Gamma_o e^{-x} = \Gamma_{po} \left[ \frac{1 + \mu - e^{-\mu x}}{\mu} \right] \tag{26}$$

The electron and proton temperatures have been assumed equal in Eq. (26), although a temperature ratio factor may be introduced in one of the exponents if necessary. With equal temperatures and number densities, the unperturbed electron-to-proton flux ratio is inversely proportional to the square root of the corresponding mass ratio, and Eq. (26) becomes

$$e^{-\mu x} + \mu \left( \frac{M}{m} \right)^{1/2} e^{-x} = 1 + \mu \tag{27}$$

If the sheath thickness is zero ($a = R$, $\mu = \infty$), Eq. (27) yields

$$x = \frac{1}{2} \ln \left( \frac{M}{m} \right)$$

or $x = 3.8$ for protons and electrons. If the sheath is infinite ($a = \infty$, $\mu = 0$), Eq. (27) becomes

$$x + \ln(1 + x) = \frac{1}{2} \ln \left( \frac{M}{m} \right)$$

and $x = 2.5$. From Eq. (27) the satellite potential increases as a increases and $\phi_R$ lies between $-3.8$ kT/e and $-2.5$ kT/e.

If the electron flux is expressed in terms of the integral flux $\psi(E)$ defined by Eq. (6), then, from Eq. (8), the equilibrium condition is

$$z\int_z^\infty \frac{\psi(E)}{E^2}\, dE = \Gamma_{po} \left[\frac{1 + \mu - e^{-\frac{\mu z}{kT}}}{\mu}\right] \tag{28}$$

where $z = xkT$. For example, the integral flux for the plasma electrons is

$$\psi(E) = \left(1 + \frac{E}{kT}\right) e^{-\frac{E}{kT}}\, \Gamma_o \tag{29}$$

so that when Eq. (29) is substituted into Eq. (28) and the integration performed, Eq. (26) is recovered. In the region of the outer radiation belt, the flux of plasma electrons to the charged satellite is negligible in comparison with the flux of high-energy trapped electrons, and the expression in Eq. (28) is more convenient.

If, for energies $E \geq z$, $\psi$ goes as $E^{-n}$, then, from Eq. (9) Eq. (28) becomes

$$\frac{\psi(z)}{n + 1} = \Gamma_{po} \left[\frac{1 + \mu - e^{-\frac{\mu z}{kT}}}{\mu}\right] \tag{30}$$

If $z_o$ and $z_\infty$ represent the values of z for zero and infinite sheath thickness, then, from Eq. (30)

$$\psi(z_o) = (n + 1)\, \Gamma_{po} \tag{31}$$

and

$$\psi(z_\infty) = \left(\frac{z_\infty}{kT}\right)(n + 1)\,\Gamma_{po} \tag{32}$$

since $z \gg kT$.

If the proton density in this region is $100/cm^3$ and the plasma temperature is $3000^\circ$ K, the unperturbed proton flux $\Gamma_{po}$ is approximately $10^7$ protons/cm$^2$/sec. Since n is usually considered to be in the range 3 to 5,[10] the value of $z_o$ is, from Eq. (31), that energy above which the unperturbed electron flux is $\sim 5 \times 10^7$ electrons/cm$^2$/sec. In the heart of the outer belt, this flux occurs above an energy of about 35 kev,[10] although occasionally a larger flux is reported.[11] An increase in flux by a factor of 100 (with the same power law) increases $z_o$ by a factor of three.

The satellite potential increases as the sheath thickness increases. From Eqs. (31) and (32), $z_\infty$ differs from $z_o$ by the factor

$$\frac{z_\infty}{z_o} = \left(\frac{kT}{z_o}\right)^{\frac{1}{n+1}} \tag{33}$$

Thus, the potential for the infinite sheath is about 1/10 that for zero sheath, and again, the potential is not strongly dependent on the thickness.

As was pointed out in Eq. (17) it is not sufficient to consider only the relative size of the sheath and satellite in order to estimate the potential as that given by the thick or thin sheath limit. The relevant criterion is whether $\mu x$ is much larger or much smaller than unity. Since $x \gg 1$, it follows from Eq. (17) that a sheath is thin (thick) when its radius is small (large) compared with $Rx^{1/2}$. If $x = 10^4$ for thick and

$x = 10^5$ for thin sheaths, then $z$ is a good approximation when $a >> 100$ R, while $z_o$ holds when $a << 300$ R.

In order to determine the sheath thickness and test the consistency of the model, it is necessary to determine the outgoing distribution functions over velocity space in order to obtain an expression for the charge density as a function of the potential. Poisson's equation may be used to eliminate the charge density, and the resulting equation for the potential must be solved subject to the necessary boundary conditions. In general, this becomes very involved mathematically, and only approximate solutions to certain special cases have been obtained.

Accurate solutions may be obtained in the limiting case where the electron and proton energies in the undisturbed plasma approach zero while their densities approach infinity in such a manner that the undisturbed flux of each is finite. In this case the ingoing and outgoing electron distribution functions and the outgoing proton distribution function vanish in the sheath. The proton density will be found to be inversely proportional to $r^2(-\phi)^{1/2}$, and a solution (in the form of an infinite series) to Poisson's equation may be found for which the potential and its first derivative vanish at the sheath edge. These solutions were first derived in the study of space-charge-limited diodes with cold emitters where the charge density between the diode elements follows directly from current and energy conservation laws.

It might seem possible to make order of magnitude estimates of the sheath thickness around a satellite, based on the cold emitting space-charge-limited diode if the emitting surface is imagined inside the sheath at a radius where the potential is sufficiently strong so that very few electrons are present and the protons are moving inward primarily in the radial direction. This approach has had some success in correlating gas

discharge measurements on Maxwellian plasmas with plane probes. Since in this case the electron density probably decreases exponentially with the Boltzmann factor, it may be neglected wherever $-e\phi$ is a few times kT, while energy considerations show that the proton velocities in these regions tend to be largely radial in direction. However, the emitter is no longer cold, since the protons have nonzero initial velocities; the emitter is not space-charge-limited, since an (unknown) electric field exists at the emitter; and the emitter current is not known. In the case of the satellite, the electron density in the sheath does not decrease as rapidly with potential, because of the presence of the high-energy electrons. It has been assumed, however, that the density of high-energy electrons in the radiation belt was at least an order of magnitude less than that of the Maxwellian electrons, and therefore their density in the sheath may be neglected. (It should be noted that this assumption requires that the steep power law dependence of the high-energy flux cannot hold for energies much below 10 kev.)

In order to avoid a net negative charge density in the sheath, the potential cannot decrease too rapidly inward from the sheath edge, and the imagined emitter surface may be well inside the sheath. At best, it might be hoped that sheath thickness calculations based on space-charge-limited diode theory give an order of magnitude estimate of the lower limit of the sheath. With these reservations, diode theory may be shown to be consistent with the estimate that a sheath is large (small) compared with $Rx^{1/2}$ when the satellite radius R is small (large) compared with the Debye length D of the undisturbed plasma (D = 40 cm for the density and temperature given).

## VII. IMPACT PARAMETER METHODS

Although extensions of the preceding calculations are most conveniently carried out through the use of the distribution functions, these preliminary results can be obtained in a somewhat more physical method through considerations of the impact parameters for grazing collision. It follows immediately from conservation of angular momentum and energy that all particles with impact parameters less than

$$R \left[\frac{E - q\phi_R}{E}\right]^{1/2}$$

where E is the particle energy outside the sheath where $\phi = 0$, and q is the particle charge, will strike the satellite of radius R and potential $\phi_R$, and the satellite cross section for collision with a particle of charge q is equal to its geometrical cross section times the factor

$$\frac{E - q\phi_R}{E}$$

These relations hold only if the sheath condition in Eq. (24) holds, and, in addition, the impact parameters lie between zero and a.

If the integral electron flux outside the sheath is $\psi(E)$, the flux of electrons with energies between E and E + dE is

$$-d\psi = -\psi'(E)\, dE$$

and the flux at the satellite becomes

$$\Gamma = -\int_{-e\phi_R}^{\infty} \left[\frac{E + e\phi_R}{E}\right] \psi'(E)\ dE \tag{37}$$

where $q = -e$. The lower limit is $-e\phi_R$, since a particle with initial energy $E < -e\phi_R$ cannot penetrate to the satellite.

Integration by parts gives

$$\Gamma = -e\phi_R \int_{-e\phi_R}^{\infty} \frac{\psi(E)}{E^2}\ dE$$

which is identical to Eq. (8), the result obtained with the use of the distribution function.

If the density in velocity space is given instead of the integral flux, the differential flux may be written directly in terms of the density. For an isotropic distribution, the number of particles with speeds between v and v + dv is proportional to $n(E)v^2\ dv$, where n(E) is the velocity space density. The differential flux is then $vn(E)v^2\ dv$, or, with the proper normalizing factor incorporated into n(E),

$$-d\psi = En(E)\ dE \tag{38}$$

This expression may be substituted into Eq. (37) to give

$$\Gamma = \int_{-e\phi_R}^{\infty} (E + e\phi_R)\ n(E)\ dE \tag{39}$$

or

$$\Gamma = \int_{0}^{\infty} En(E - e\phi_R)\ dE \tag{40}$$

For the Maxwellian distribution,

$$n(E - e\phi_R) = e^{\frac{e\phi_R}{kT}} n(E)$$

and

$$\Gamma = e^{\frac{e\phi_R}{kT}} \int_0^{\infty} En(E)\, dE$$

$$= e^{\frac{e\phi_R}{kT}} \Gamma_o \tag{41}$$

which is identical to Eq. (5), the result obtained from the distribution function.

The potential is attractive for protons, and all protons incident on the sheath with initial energies between zero and some critical energy $E_c$ strike the satellite. The critical energy is that energy which gives an impact parameter of a for a grazing collision,

$$a^2 = R^2 \left[\frac{E_c - e\phi_R}{E_c}\right]$$

or,

$$E_c = \frac{R^2 e\phi_R}{R^2 - a^2} \tag{42}$$

If $n_p(E)$ is the density for protons, then

$$\Gamma_p = \frac{a^2}{R^2} \int_0^{E_c} E n_p(E)\, dE + \int_{E_c}^{\infty} (E - e\phi_R)\, n_p(E)\, dE \qquad (43)$$

where the first integral involves a constant cross section times the differential flux, and the second involves the energy-dependent cross section times the same flux. The integration may be carried out for a Maxwellian distribution with the result given by Eq. (15). As before, the result in Eq. (43) is true only if the potential satisfies Eq. (24).

If the critical energy $E_c$ is much greater than the characteristic plasma energy kT, nearly all the incident protons will strike the satellite. Since from Eq. (42), the ratio of $E_c$ to kT is equal to $\mu x$, the strong potential condition in Eq. (17) that $\mu x \gg 1$ follows.

## VIII. CONCLUSIONS

Most of the difficulties and inadequacies of ordinary probe theory are well known. Even in the simplest theory, that of the idealized sheath, inconsistencies are encountered in the methods for the solution of a basic problem: the determination of the potential of the floating probe in a collisionless, homogeneous, isotropic plasma. In this report the problem was treated from the point of view of distribution functions, and the inconsistencies are associated with the assumed boundary condition for the Boltzmann equation at the edge of the sheath. If the inconsistencies are neglected, the results show that the potential is almost independent of the sheath thickness.

Recently, floating probe theories have been adapted to the problems of satellite charge-up and electric drag. The results of this report show that even in the presence of suprathermal flux distributions, such as the Van Allen flux, the satellite potential is a weak function of the sheath thickness.

Diode theory is a special limiting case of idealized sheath theory, and its use in sheath thickness claculations is subject to considerable inaccuracies. Because of the weak dependence of the potential on the thickness, the use of diode theory leads to little error in satellite potential calculations, but its use in more sensitive areas, such as in electric drag claculations, will likely lead to significant errors.

To determine the satellite or probe equilibrium potential, it is convenient to calculate first the positive and negative currents to the probe as functions of the probe potential, and then to equate the currents and solve for the potential. If $\psi$ (E) is the unperturbed flux of negative

particles with energies greater than E, then the flux $\Gamma$ of such particles to an absorbing, negatively charged spherical probe with potential $\phi_R$ is given by

$$\Gamma = z \int_z^\infty \frac{\psi(E)}{E^2} \, dE \tag{8}$$

where $z = -e\,\phi_R$, and -e is the charge on the particle. The flux given by Eq. (8) is a consequence of the idealized sheath theory, and, although it does not depend on either the probe or sheath dimensions, it is necessary to assume that the potential throughout the sheath region lies between zero and $\phi_R$. However, if the unperturbed plasma is electrically neutral, the theory predicts a positive sheath about a negative probe, and the condition is satisfied. In fact, a positive sheath means the potential is a monotonic function of the radial distance throughout the sheath, and lies between zero and a limiting potential which varies as 1/r between $\phi_R$ at the inside edge, and zero at the outer boundary.

If the unperturbed negative paritcles have a Maxwellian distribution, Eq. (8) reduces to the well-known expression

$$\Gamma = \psi(0) \exp\left(\frac{e\phi_R}{kT}\right)$$

Another example, of interest in connection with the trapped electrons in the earth's radiation belt, is a flux $\psi(E)$ which goes as $E^{-n}$ for $E > z$, and for which Eq. (8) gives

$$\Gamma = \frac{\psi(z)}{n + 1} \tag{9}$$

The positive particles are moving in an attractive field and a general expression for the flux to the probe, corresponding to that for the negative

particles given by Eq. (8), cannot be obtained. An upper limit to the positive flux can be obtained with the assumption that all trajectories are filled for which the particle energy and angular momentum at the outer boundary of the sheath are equal to the energy and angular momentum at the inner edge. If the unperturbed velocity distribution of positive particles is not restricted, this assumption implies that the potential throughout the sheath lies below a limiting potential which varies as 1 $r^2$ between $\phi_R$ at the inside edge and zero at the outer boundary. If this condition on the potential holds, and if the positive particles have a Maxwellian velocity distribution, the positive flux to the probe is given by the Mott-Smith expression shown in Eq. (15).

If the high-energy electron flux in the more intense regions of the earth's radiation belt is $\sim 5 \times 10^7$ electrons/$cm^2$/sec above 35 kev, with the flux varying as $E^{-4}$, and if the protons have a Maxwellian distribution at $\sim 3000^\circ$ K with a density of $\sim 100/cm^3$, the satellite equilibrium potential will, from Eqs. (9) and (15), be $\sim$ 10,000 volts negative. This potential should be reduced (in magnitude) by a factor of $\sim 3$ for a satellite with a diameter much smaller than 1 meter, and should be multiplied by $\sim 3$ for a satellite much greater than 1 meter in diameter.

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